EN5101 Digital Control Systems

Pulse Response

Digital Approximations

Stability

Prof. Rohan Munasinghe

Dept of Electronic and Telecommunication Engineering

University of Moratuwa

Transfer Function of a Discrete System

transfer function in sampled systems

$$y(k) + a_1 y(k-1) + \cdots + a_n y(k-n) = b_0 u(k) + b_1 u(k-1) + \cdots + b_m u(k-m)$$

using shifting theorem in z-transform

$$Y(z) + a_1 z^{-1} Y(z) + \cdots a_n z^{-n} Y(z) = b_0 U(z) + b_1 U(z) z^{-1} + \cdots + b_m z^{-m} U(z)$$

$$(1 + a_1 z^{-1} + \cdots + a_n z^{-n}) Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}) U(z)$$

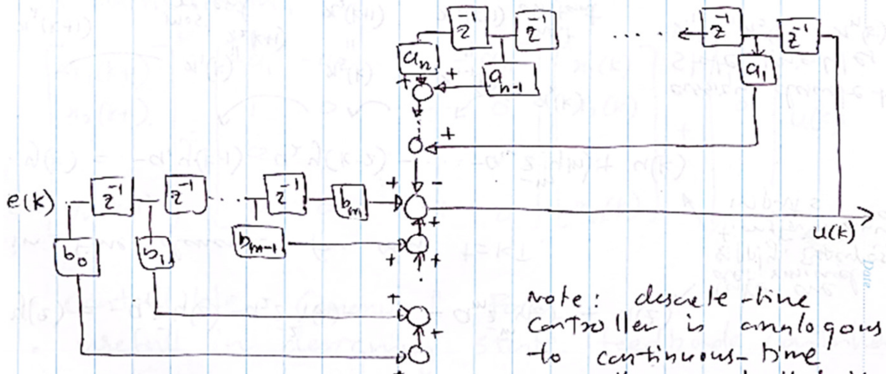
$$D(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}$$

Let's draw the controller structure

Structure of a Discrete Controller (Tra Fn)

$$e(k) \to \boxed{C(z)} \to u(k)$$

$$U(z) + a_1 z^{-1} U(z) + \cdots + a_n z^{-n} U(z) = b_0 E(z) + b_1 E(z) z^{-1} + \cdots + b_m z^{-m} E(z)$$

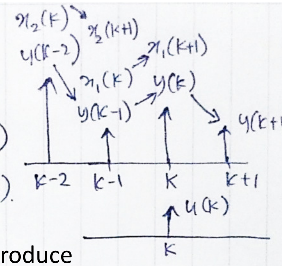


Note: discrete-time controller is analogous to continuous-time controller except that 1/s (integrators) are replaced by delay operators (z^{-1}).

Tra Fn to State Space Model in DT

Example

$$\frac{Y(z)}{U(z)} = \frac{2}{1 + 2z^{-1} + 3z^{-2}}$$



$$Y(z) = -2z^{-1}Y(z) - 3z^{-2}Y(z) + 2U(z)$$

z^{-1}

$$y(k) = -2y(k-1) - 3y(k-2) + 2u(k)$$

$x_1(k+1)$ $\leftarrow$ $x_1(k)$, $x_2(k+1)$ $\leftarrow$ $x_2(k)$ — 1. Introduce state variables

State Equations

$$x_1(k+1) = -2x_1(k) - 3x_2(k) + 2u(k)$$

$$x_2(k+1) = x_1(k)$$

2. Write state equations

SSM

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$

$$y(k) = x_1(k+1)$$

3. Put it in matrix form

Tra Fn to State Space Model in DT

$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_n z^{-n} Y(z) + U(z)$

determined by the Controller $E(z)$

in time-domain for any $t = kT$.

$y(k) = -a_1 y(k-1) - a_2 y(k-2) - \cdots - a_n y(k-n) + u(k)$

$x_1(k+1)$ — output equation

$x_1(k) = x_2(k+1)$ most recent value

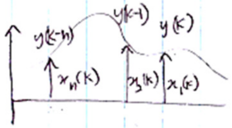
$x_2(k) = x_3(k+1)$

$x_{n-1}(k) = x_n(k+1)$ most distant value

$x_n(k)$

State equations

assign discrete time State variables $x_1(k), x_2(k), \ldots x_n(k)$



we can write state equations as follows

$x_1(k+1) = -a_1 x_1(k) - a_2 x_2(k) + \cdots - a_n x_n(k) + u(k)$

$x_2(k+1) = x_1(k)$

$x_3(k+1) = x_2(k)$

$\vdots$

$x_n(k+1) = x_{n-1}(k)$

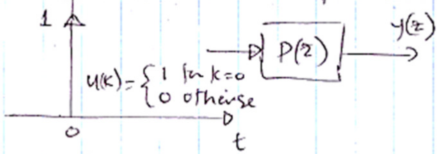
In matrix form,

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(k)$$

- Controllable Canonical Form.
- usefull in designing state feedback Controller

Unit Pulse Response

unit pulse response (first order system)



$U(z) = \sum_{k=0}^{\infty} u(k) z^{-k} = u(0) z^{-0} = 1$

$Y(z) = P(z)\, U(z)$

$y(k) = Z^{-1}[P(z) \cdot \underbrace{U(z)}_{=1}]$

$= Z^{-1}[P(z)]$ Pulse response of a System is the inverse z-trans fon of the System transfer function.

Eg: $P(z) = \frac{b_1 z}{z - a_1}$ first order (single pole)

pulse response

$y(k) = Z^{-1}[P(z)]$. Recall $Z[a^k] = \frac{z}{z-a}$

$= Z^{-1}\left[b_1 \frac{z}{z - a_1}\right]$

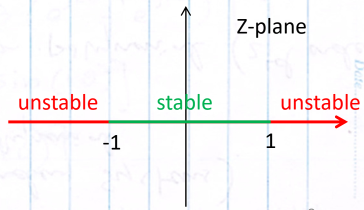
$= b_1 a_1^k$ exponential decay for $|a_1| < 1$

" rise for $|a_1| > 1$

non oscillating if a_1 is +ve

oscillating if a_1 is -ve

sign \ size: a_1	$\|\cdot\| > 1$	$\|\cdot\| < 1$
-ve	oscillating growing (unstable)	oscillating decaying (stable)
+ve	non oscillating growing (unstable)	non oscillating decaying (stable)



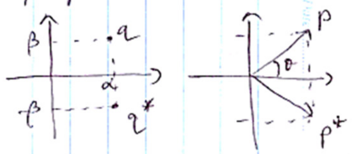
Unit Pulse Response – 2nd Order System

$$P(z) = z\,\frac{\overbrace{N(z)}^{\text{numerator polynomial}}}{\underbrace{z^2 + a_1 z + a_2}_{\text{denominator polynomial (2nd order)}}}$$

general 2nd order system has two poles, a pair of complex conjugate poles, thus can be resolved by partial fraction as follows

$$P(z) = z\left[\frac{q}{z-p} + \frac{q^*}{z-p^*}\right]$$ where q, q^* ; p, p^* complex conjugate pairs

$$= z\left[\frac{\alpha+j\beta}{z-re^{j\theta}} + \frac{\alpha-j\beta}{z-re^{-j\theta}}\right]$$



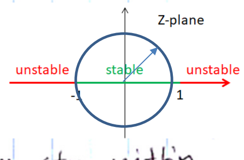
unit pulse response

$$u(z)=1 \rightarrow \boxed{P(z)} \rightarrow y(z)$$

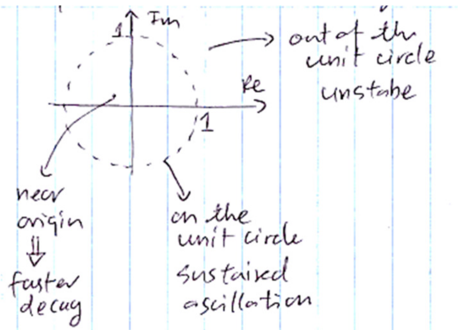
System poles $z = re^{\pm j\theta}$
$= r(\cos\theta \pm j\sin\theta)$

$$y(k) = Z^{-1}\left[P(z)\right]$$
$$= Z^{-1}\left[(\alpha+j\beta)\frac{z}{z-re^{j\theta}} + (\alpha-j\beta)\frac{z}{z-re^{-j\theta}}\right]$$
$$= (\alpha+j\beta)\,Z^{-1}\left[\frac{z}{z-re^{j\theta}}\right] + (\alpha-j\beta)\,Z^{-1}\left[\frac{z}{z-re^{-j\theta}}\right]$$
$$= (\alpha+j\beta)(re^{j\theta})^k + (\alpha-j\beta)(re^{-j\theta})^k$$
$$= r^k\left[\alpha\left(e^{jk\theta} + e^{-jk\theta}\right) + j\beta\left(e^{jk\theta} - e^{-jk\theta}\right)\right]$$
$$= r^k\left[\alpha\, 2\cos(k\theta) + j\beta\, 2j\sin(k\theta)\right]$$
$$= \underbrace{2r^k}_{\text{magnitude}}\left[\underbrace{\alpha\cos(k\theta) - \beta\sin(k\theta)}_{\text{oscillation}}\right]$$

System is stable if $|r| < 1$ ⇒ poles stay within unit circle



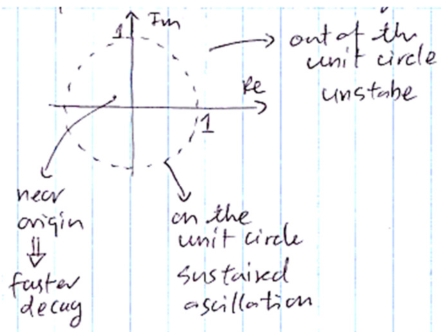
Pole Radius and Decay rate



Response magnitude = $2r^k$

Quiz: Determine the number of samples for the response to decay down to 1% of the initial magnitude

Pole Radius and Decay rate



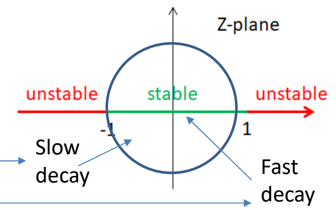
Response magnitude = $2r^k$

samples to decay to 1% of the initial value

magnitude at $k=0$ $= 2$
" " $k=n$ $= 2r^n$

$$\frac{|\ |_{k=n}}{|\ |_{k=0}} = r^n$$
$$0.01 = r^n$$
$$\frac{\log 0.01}{\log r} = n$$

r	# Samples to 1%
0.9	43
0.8	21
0.6	9
0.4	5

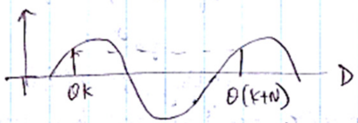


Pole Angle and Oscillation

\# samples per period $= N$ then

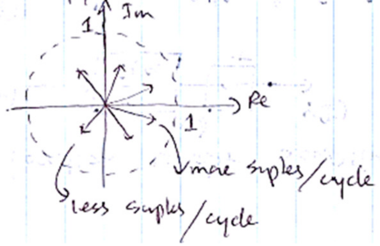
$\cos(\theta k) = \cos\{\theta(k+N)\} \Rightarrow \theta N = 2\pi$

$N = \frac{2\pi}{\theta}$



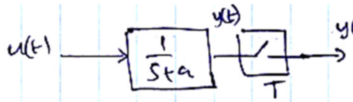
θ smaller → more samples per period.

θ bigger → less " " "



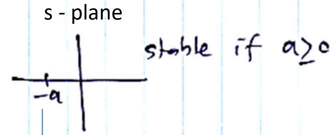
Slow decay

Effect of Sampling on Pole Locations



1st order System

$\frac{Y(s)}{U(s)} = \frac{1}{s+a}$ pole at $s=-a$



$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+a}U(s)\right]$

unit impulse response $U(s)=1$.

$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+a}\right]$

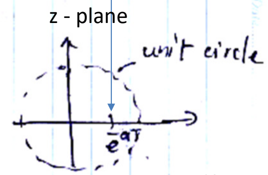
$= e^{-at}$

Continuous to discrete pole mapping

In discrete mode

$y(k) = y(t)\big|_{t=kT}$ $k=0,1,2\ldots$

$= e^{-aTk}$



Z-tr $Y(z) = \frac{z}{z-e^{-aT}}$ pole at $z = e^{-aT}$

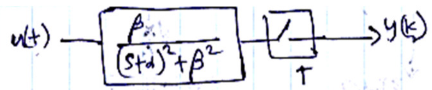
Stable Or Unstable?

- CT system $\frac{1}{s+a}$ is stable $\forall$ a>0. Then, the sampled system has to be stable as well
- As per the corresponding DT system $\frac{z}{z-e^{-aT}}$, stability criterion is

$$e^{-aT} \leq 1$$
$$\ln(e^{-aT}) \leq \ln 1$$
$$-aT \leq 0$$

which is true for any sampling interval T

Second Order System

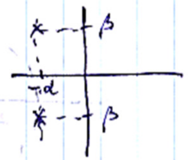


$\frac{Y(s)}{U(s)} = \frac{\beta}{(s+\alpha)^2+\beta^2}$ $\Delta(s) = (s+\alpha)^2+\beta^2 = 0 \Rightarrow s_1 s_2 = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+\beta^2)}}{2}$

$= -\alpha \pm j\beta$

unit impulse response $U(s)=1$

$y(t) = \mathcal{L}^{-1}\left[\frac{\beta}{(s+\alpha)^2+\beta^2}\right] = e^{-\alpha t}\sin(\beta t)$ stable if $\alpha \geq 0$



In discrete mode

$y(k) = y(t)\big|_{t=kT}$ $k=0,1,2\ldots$

$= e^{-\alpha kT}\sin(\beta kT)$ $k \geq 0$

table Z

$Y(z) = \frac{z e^{-\alpha T}\sin(\beta T)}{z^2 - 2z e^{-\alpha T}\cos(\beta T) + e^{-2\alpha T}}$

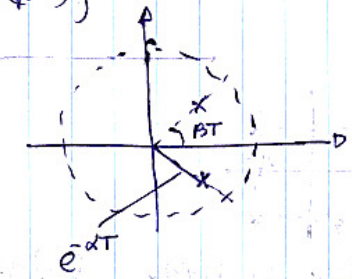
$\Delta(z) = z^2 - 2z e^{-\alpha T}\cos(\beta T) + e^{-2\alpha T} = 0$

$$z_1, z_2 = 4e^{-\alpha T}\cos(\beta T) \pm \sqrt{4e^{-2\alpha T}\cos^2(\beta T) - 4e^{-2\alpha T}} \;/2$$

$$= e^{-\alpha T}\left[\cos(\beta T) \pm j\sin(\beta T)\right]$$

$$= e^{-\alpha T}\, e^{\pm j\beta T}$$

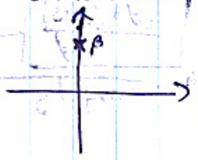
$$= e^{(-\alpha \pm j\beta)T}$$

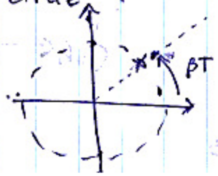


Case I $\alpha = 0$ radius $= 1 = e^{-\alpha T}$

Sustained oscillations

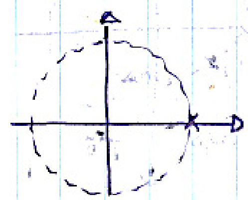
on the unit circle





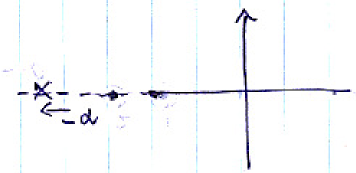
case II $\alpha = 0$, $\beta = 0$ radius $= 1 = e^{-\alpha T}$, angle $\beta T = 0$

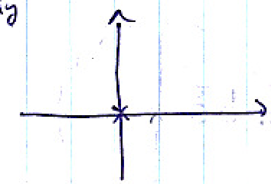
rigid body dynamics



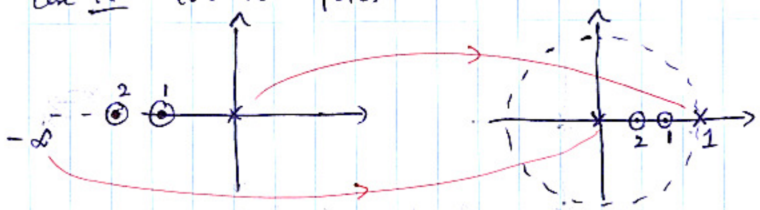
case III $\alpha \to -\infty$ $\beta = 0$ radius $= e^{-\alpha T} \Rightarrow 0$

Pole (real) at infinity

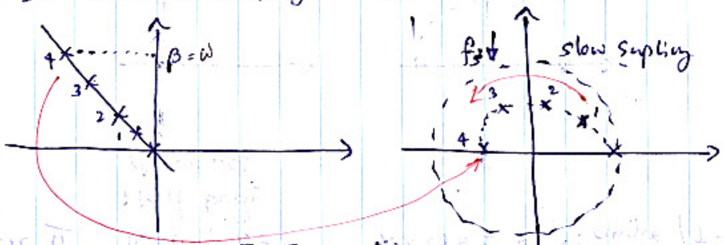




case IV two real poles



case V Constant damping ratio



$\omega T = \beta T = \pi$

one sample / half period $\Rightarrow$ Nyquist limit

Eg: Find the discrete time pole-locations of $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for $\zeta = 0.6$, $\omega_n = 5$ rad/s.

- check stability against sampling interval T ($|\,| < 1$)
- check aliasing " " " T ($\angle < 180°$)

$$G(s) = \frac{5^2}{s^2 + 2\times 0.6\times 5\, s + 5^2} = \frac{25}{s^2 + 6s + 25}$$

$$\Delta(s) = s^2 + 6s + 25 = 0 \Rightarrow \text{poles } s_1 s_2 = \frac{-6 \pm \sqrt{36 - 4\times 25}}{2} = -3 \pm j4$$

ω_n ω_d

discrete-time poles $z_1 z_2 = e^{(s_1, s_2)T} = e^{(-3 \pm j4)T} = e^{-3T} \cdot e^{\pm j4T}$

$|\,|$ mag $\angle$ ang

Alternatively

$z_1 z_2 = e^{-3T}\left(\cos 4T \pm j\sin 4T\right)$

$\omega T_P = 2\pi$

$T_P = \frac{2\pi}{\omega} = 1.5$

Stability

$|e^{-3T}| < 1$ This is always true for any $T > 0$

note: A stable $G(s)$ has stable $Z(s)$ under any T. however, slow sampling may cause aliasing in the feedback loop which will deteriorate the response.

for sampling at 4 Hz $T = 0.25$ s

$z_1 z_2 = 0.255 \pm j0.398$